ENERGY TRANSFER BY SPECTRAL-LINE RADIATION

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The transfer of radiant energy in moving media is described by a set of integrodifferential equations. The integral term in these equations necessitates an enormous number of calculations. In three-dimensional or even axisymmetric flows it is necessary to evaluate a quadruple integral at each point of space.

In this paper we consider the case of local thermodynamic equilibrium, assuming that the spectral lines are both temperature- and pressure-shifted. The absorption coefficient of the gas is taken in idealized form, which substantially simplifies radiative heat transfer and enables us to integrate explicitly the transfer equation. In the case of spherical symmetry, which is considered as an example, the simplification is greater still and leads to differential equations without the integral terms. If radiation and absorption occur in a finite number of spectral lines, the resultant amounts of heat received by a particle due to radiation can be found by simple summation over the characteristic line frequencies. This representation of the absorption coefficient can be used as a basis for numerical methods.

1. The motion of a radiating ideal gas in the case of local thermodynamic equilibrium is described by the following system:

$$\begin{gathered}\frac{\partial\rho}{\partial t}+\operatorname{div}\rho\mathbf{v}=0,\quad \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\,\mathbf{v}+\frac{1}{\rho}\nabla p=0\\ \frac{\partial}{\partial t}\left(\frac{\rho v^2}{2}+\rho i\right)+\operatorname{div}\rho\mathbf{v}\left(\frac{v^2}{2}+H\right)=Q\\ p=p\,(\rho,T),\qquad i=i\,(\rho,T),\qquad H=H\,(\rho,T)\end{gathered}\tag{1.1}$$

where t is time, v is the velocity vector, ρ is density, p is pressure, T is temperature, i is the internal energy per unit mass, and H is the enthalpy per unit mass.

The radiant energy flux per unit volume in the case of continuous functions is [1]

$$Q=\int_0^\infty\int_0^{2\pi}\int_0^\pi \rho\varkappa_\nu J_\nu\sin\vartheta\,d\vartheta\,d\varphi\,d\nu-4\pi\rho\int_0^\infty\varkappa_\nu B_\nu\,d\nu \tag{1.2}$$

$$\frac{1}{\rho\varkappa_\nu}\frac{\partial J_\nu}{\partial s}=J_\nu-B_\nu \tag{1.3}$$

where the last differential corresponds to the first integral, the next-to-last differential corresponds to the second integral, and so on. The quantities s, θ, and φ are polar coordinates ($s \geq 0$) with origin ($s = 0$) at a given point of three-dimensional space; ν is the frequency; $\varkappa_\nu$ is the absorption coefficient at the particular frequency; J_ν is the intensity of the radiant energy flux at frequency ν along the ray θ = const, φ= const in the direction toward a given point; and B_ν is the ratio of the emission and absorption coefficients at the frequency ν. The intensity J_ν is determined using the transfer equation (1.3) and the boundary condition which depends on the particular problem. The signs in Eq. (1.3) and in the corresponding equations in [1] are different because here we are considering the radiant energy flux in the direction of decreasing s, whereas in [1] the flux is in the opposite direction. In the case of local thermodynamic equilibrium

$$B_\nu=\frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/kT)-1}$$

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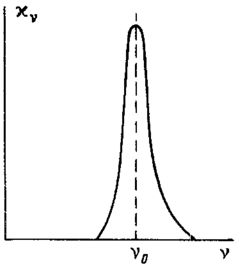

Fig. 1

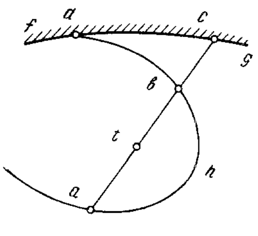

Fig. 2

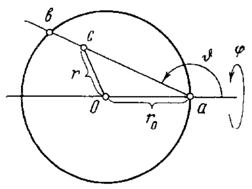

Fig. 3

where h is Planck's constant, k is Boltzmann's constant, and c is the speed of light.

The quantity Q gives the energy received by a particle due to radiation. The first term on the right-hand side of Eq. (1.2) represents the quantity of radiant energy absorbed per unit volume. This is absorbed from the energy which arives along all directions with intensity J_ν. Integration with respect to the frequency yields the total amount of absorbed energy. The second term on the right-hand side of Eq. (1.2) gives the total amount of energy emitted per unit volume.

The transfer equation given by Eq. (1.3) shows that the intensity J_ν in the direction of decreasing s is increased by emission (the term B_ν) and is reduced by absorption (the term J_ν). If there is no absorption, J_ν is determined by simple integration of $-\rho \varkappa_\nu J_\nu$ with respect to s. On the other hand, if there is no emission ($J_\nu = 0$), the homogeneous equation given by Eq. (1.3) determines the change in the intensity J_ν due to absorption alone. These elementary remarks will be used below.

The characteristic form of the dependence of $\varkappa_\nu$ on ν in the case of line emission is shown in Fig. 1. Moreover, $\varkappa_\nu$ is also a function of temperature and pressure, i.e., $\varkappa_\nu = \varkappa(\nu, p, T)$. The dependence on p and T changes the shape of the curve (Fig. 1) and shifts it toward other frequencies. A review of line emission is given in [2].

In the case of narrow lines, i.e., short ν-interval in which $\varkappa_\nu \neq 0$, the simplest approximate approach to the calculation of the radiative energy transfer is obtained by writing

$$\varkappa_\nu = K(p, T)\ \delta(\nu - \nu_0) \tag{1.4}$$

where $\delta(\nu - \nu_0)$ is the Dirac δ-function, and $\nu_0 = n(p, T)$ is the frequency corresponding to the center of gravity of the area between the $\varkappa_\nu$ curve and $\varkappa = 0$ in Fig. 1 (this choice of the function θ is made in order to be specific). At the same time,

$$K(p, T) = \int_0^\infty \varkappa_\nu \, d\nu$$

We assume that at least one of the derivatives $\partial n/\partial p$, $\partial n/\partial T$ is not zero at each point.

The validity of the above form of $\varkappa_\nu$ and the error introduced by this representation is not investigated here.

The transfer equation given by Eq. (1.3) can be formally integrated if we suppose that all the functions in this expression, except J_ν, are known [1]. At the point which we are considering, the quantity s is zero (s is the radial distance in spherical polars). Suppose that the radial distance to the boundary of the region is $S(\theta, \varphi)$. It follows that if we require the intensity J_ν, arriving at s = 0, integration yields ($\varepsilon > 0$)

$$J_\nu^+(0) = J_\nu(S)\, e^{-\tau_\nu^+(0, S)} + \lim_{\varepsilon \to 0} \int_\varepsilon^S B_\nu e^{-\tau_\nu^+(0, s')} \varkappa_\nu \rho \, ds' \tag{1.5}$$

$$\tau_\nu^+(0, s') = \lim_{\varepsilon \to 0} \int_\varepsilon^{s'} \varkappa_\nu \rho \, ds'' \tag{1.6}$$

The superscript + refers to the limit $\varepsilon = +0$. The quantity $J_\nu(S)$ in Eq. (1.5) is the intensity of radiation of frequency ν which travels away from the boundary of the region in the direction defined by and φ. The quantity τ_ν is called the optical thickness of the layer (0,s') at the frequency ν.

If the boundary lies at infinity in the direction θ, φ, and the radiant energy flux propagating from infinity toward the given point is zero, then

$$J_\nu^+(0) = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty B_\nu e^{-\tau_\nu^+(0, s')} \varkappa_\nu \rho \, ds' \tag{1.7}$$

2. There is no difficulty in using Eqs. (1.5) and (1.3) when the absorption coefficient is continuous. When it is given by a δ-function, the absorption of radiant energy in the neighborhood of a given point must be found by calculating J_ν separately before and after the flux passes this point. This requires a preliminary determination of the optical thickness

$$\tau_1 = \tau_\nu^+(0, S), \quad \tau_2 = \tau_\nu^-(0, S) = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{S} \varkappa_\nu \rho \, ds'' \tag{2.1}$$

Let us denote the point under consideration by a. In the space which we are considering we can draw a plane through the ray ac leaving point a with given θ and φ. Let us suppose that this plane lies in the plane of Fig. 2. The line of intersection of the boundary of the region with our plane will be denoted by fg. Suppose further that the frequency $\nu_0 = n(p, T)$ corresponds to p and T at a. Let us isolate a surface in the three-dimensional region in which the frequency ν_0 is equal to the frequency n at a. The line of intersection of this surface with our plane will be denoted by $a$hbd. (If ν_0 is independent of pressure, $a$hbd will be an isotherm). We assume henceforth that the straight line ac cuts the surface of constant frequency ν_0 at two points. Generalization to more complicated cases presents no difficulty, but the analysis is more difficult to interpret.

Gas in the neighborhood of point a can absorb radiant energy only at the frequency ν_0 corresponding to p_a and T_a. Therefore, we shall confine our attention to the emission at this frequency when we consider point a. For the sake of simplicity, the corresponding subscript is not written out explicitly. We can now see the principle which follows from the chosen representation of the absorption coefficient given by Eq. (1.4) and greatly facilitates the analysis of energy transfer processes: radiant-energy transfer inside the region occurs only between points on surfaces with the same frequency ν_0.

Let us first calculate τ_1 which is given by Eqs. (2.1) and (1.6). If at the point t on the straight line ac the function n(p, T) reaches a maximum or a minimum, then by using Eqs. (1.4) and (1.6) we can rewrite Eq. (2.1) in the form

$$\tau_1 = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{s_t} F \, ds'' + \int_{s_t}^{S} F \, ds'' \tag{2.2}$$

$$F = K(p, T) \rho \delta [n(p, T) - \nu_0].$$

It follows from this formula that F = 0 for $s \neq 0$ and $s \neq s_b$, since at all other points emission and absorption occur at other frequencies, and $n(p, T) \neq \nu_0$.

The integration is carried out along the straight line ac on which $p = p(s)$, $T = T(s)$, and $\rho = \rho(s)$, and, consequently, n is a function of s. Whenever n is a monotonic function of s, we can write

$$ds'' = m \, dn, \quad m = \left(\frac{\partial n}{\partial T} \frac{dT}{ds} + \frac{\partial n}{\partial p} \frac{dp}{ds} \right)^{-1} \tag{2.3}$$

The quantities T and p are functions of time and of space coordinates. The total derivatives in the last equation represent differentiation along ac at given time. Substituting Eq. (2.3) in Eq. (2.2), and remembering that F = 0 for $s \in (0, s_t)$, we obtain

$$\tau_1 = \int_{n_t}^{n_c} mK\rho\delta(n - \nu_0) \, dn$$

where m, K, and ρ are now regarded as functions of n along tc.

Earlier we assumed that the straight line ac cuts the $n = \nu_0$ surface only at a and b. If there are no other surfaces with $n = \nu_0$ in space, then the equality $n = \nu_0$ within the interval (0, s) is reached only at b. Using the last equation, we have

$$\tau_1 = (mK\rho)_b \, \text{sign} \left(\frac{dn}{ds} \right)_b = (|m| K\rho)_b \tag{2.4}$$

where we have used the fact that

$$\int_0^\infty \delta(n-\nu_0)\,dn = 1$$

The derivative dn/ds is evaluated along the straight line ac.

Let us now find the quantity τ_2 given by Eq. (2.1). If we recall Eq. (2.2), we may write

$$\tau_2 = \lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon} \varkappa_\nu \rho\, ds'' + \tau_1$$

As before, we can write ds" in the form of Eq. (2.3) and, using Eq. (1.4), we find that

$$\tau_2 = \lim_{\varepsilon\to 0}\int_{n_{-\varepsilon}}^{n_\varepsilon} mK\rho\delta(n-\nu_0)\,dn + \tau_1 \tag{2.5}$$

where $n_{-\varepsilon}$ and n_ε are the frequencies n(p, T) corresponding to the values of p and T for $s = -\varepsilon$ and $s = \varepsilon$ on the straight line ac. If the functions p and T are continuous along this line, ν_0 lies between $n_{-\varepsilon}$ and n_ε. Hence, the integral in the last equation can be evaluated, and Eq. (2.5) yields

$$\tau_2 = (|m|\,K\rho)_a + \tau_1 \tag{2.6}$$

When we calculate the total radiant energy flux Q we need to know J_ν and, consequently, τ_2 as well, for all the values of θ and φ used in Eq. (1.2). The quantity m_a in the formula can conveniently be transformed to a form which does not contain derivatives with respect to s in different directions. Using Eq. (2.3), we can write m_a (θ, φ) in the form

$$m_a = \left[(\nabla T\cdot \mathbf{s})\frac{\partial n}{\partial T} + (\nabla p\cdot \mathbf{s})\frac{\partial n}{\partial p}\right]^{-1} \tag{2.7}$$

where s is a unit vector defined by θ, φ and drawn in the direction of increasing s; and the parentheses on the right-hand side denote scalar products.

3. Let us now determine the intensity J_ν at the frequency ν_0, which arrives at point a. In the case of bounded or infinite regions, this can be found from Eq. (1.5) or Eq. (1.7). The second term on the right-hand side of Eq. (1.5) can be written as the sum of three integrals:

$$\lim_{\varepsilon\to 0}\int_\varepsilon^S B_\nu e^{-\tau_\nu^+(0,\, s')}\varkappa_\nu\rho\, ds' = \lim_{\varepsilon\to 0}\int_\varepsilon^{s_b-\varepsilon} B_\nu e^{-\tau_\nu^+}\varkappa_\nu\rho\, ds' - \lim_{\varepsilon\to 0}\int_{\tau_\nu^+(0,\, s_b-\varepsilon)}^{\tau_\nu^+(0,\, s_b+\varepsilon)} B_\nu\, de^{-\tau_\nu^+} + \lim_{\varepsilon\to 0}\int_{s_b+\varepsilon}^{S} B_\nu e^{-\tau_\nu^+}\varkappa_\nu\rho\, ds'$$

The integrands in the first and third terms on the right-hand side of this equation contain the factor $\varkappa_\nu$ which is zero for $s' \in (0, s_b)$ and $s' \in (s_b, S)$. The other factors are bounded. It follows that the first and third terms are zero, and if we evaluate the integrals, we obtain

$$-B_{\nu b}\lim_{\varepsilon\to 0}\left[e^{-\tau_\nu^+(0,\, s_b+\varepsilon)} - e^{-\tau_\nu^+(0,\, s_b-\varepsilon)}\right]$$

However, $\tau_\nu^+(0, s_b+\varepsilon) = \tau_\nu^+(0, S) = \tau_1$, since $\varkappa_\nu$ is zero for $s \in (s_b, S]$. Moreover, τ_ν^+ $(0, s_b-\varepsilon) = 0$, since the quantity $\varkappa_\nu$ for $s \in (0, s_b)$ is also zero. Finally, by comparing the last results with Eq. (1.5), we conclude that

$$J_\nu^+(0) = J_\nu(S)\, e^{-\tau_1} + B_{\nu b}(1 - e^{-\tau_1}) \tag{3.1}$$

where τ_1 is given by Eq. (2.4).

In the case of an unbounded region, and in the absence of radiation at infinity, Eq. (1.7) shows that

$$J_\nu^+(0) = B_{\nu b}\,(1 - e^{-\tau_1}) \tag{3.2}$$

where we have used the fact that $\tau_\nu^+(0, \infty) = \tau_1$ subject to the condition that the ray ac intersects the surface with constant $n(p, T) = \nu_0$ only once.

The first term on the right-hand side of Eq. (3.1) has the usual form and shows the reduction in the radiant energy flux with the optical thickness of the layer. The second term on the right-hand side of Eq. (3.1) and the right-hand side of Eq. (3.2) gives the resultant intensity J_ν produced by an element of the surface with $n = \nu_0$.

Let us now determine the radiant-energy flux passing through point a on the assumption that there is no emission at a. In the ε-neighborhood of a, the intensity J_ν will be determined by the homogeneous variant of Eq. (1.3):

$$\frac{\partial J_\nu}{\partial s} = \rho \varkappa_\nu J_\nu$$

If we integrate along ac, we obtain

$$J_\nu^- (0) = \lim_{\varepsilon \to 0} J_\nu (-\varepsilon) = J_\nu^+ (0)\, e^{-(\tau_2 - \tau_1)}$$

where $\tau_2 - \tau_1$ is readily found from Eq. (2.6):

$$\tau_3 = \tau_2 - \tau_1 = (|m| \; K\rho)_a \tag{3.3}$$

The quantity by which the intensity has been reduced on passing through a is given by the difference $\Delta J_\nu (0) = J_\nu^+ (0) - J_\nu^- (0)$. For a bounded region we have

$$\Delta J_\nu (0) = (1 - e^{-\tau_3}) \left[J_\nu (S)\, e^{-\tau_1} + B_{\nu_b} (1 - e^{-\tau_1}) \right] \tag{3.4}$$

and when $J_\nu (S) = 0$

$$\Delta J_\nu (0) = B_{\nu b} (1 - e^{-\tau_1}) (1 - e^{-\tau_3}) \tag{3.5}$$

where we have used Eqs. (3.1) and (3.2).

The quantity ΔJ_ν (0) is the intensity loss due to absorption at a.

4. Let us now calculate the total radiant-energy flux Q_1 absorbed per unit volume. Let us suppose, to begin with, that the absorption coefficient is continuous. The loss of intensity due to absorption in a given direction over a path length ds at frequency ν is $dJ_\nu = J_\nu \rho \varkappa_\nu ds$. The loss of intensity in a cylindrical element of cross section $d\sigma$ and height ds is $J_\nu \rho \varkappa_\nu ds d\sigma$ or $J_\nu \rho \varkappa_\nu dw$, where dw is the volume element. The loss of intensity per unit volume is $J_\nu \rho \varkappa_\nu$. To obtain the total amount absorbed per unit volume we must integrate with respect to frequency and solid angle. As a result, we obtain the first term on the right-hand side of Eq. (1.2)

$$Q_1 = \int_0^\infty \int_0^{2\pi} \int_0^\pi \rho \varkappa_\nu J_\nu \sin \vartheta \, d\vartheta \, d\varphi \, d\nu$$

Let us now find the quantity Q_1 for an absorption coefficient of the form given by Eq. (1.4). The change of intensity in the cylindrical element of cross section $d\sigma$ and height ds in the direction of $-$s at frequency ν_0 is $\Delta J_\nu(0) d\sigma$. At all frequencies n(p, T) corresponding to values of p and T in the cylindrical element, the change of flux is $\Delta J_\nu (0) \, |dn/ds| \, d\sigma \, ds$ or $\Delta J_\nu (0) \, |m|^{-1} dw$, where m is given by Eq. (2.3). The loss of flux per unit volume is $\Delta J_\nu |m|^{-1}$. To obtain the total flux absorbed per unit volume we must integrate with respect to the solid angle:

$$Q_1 = \int_0^{2\pi} \int_0^\pi \Delta J_\nu (0) \, | m |^{-1} \sin \vartheta \, d\vartheta \, d\varphi$$

It is not necessary to integrate with respect to frequency in this case because one definite frequency corresponds to each point in space.

The amount of energy Q_2 emitted per unit volume is calculated by integrating the second term on the right-hand side of Eq. (1.2). This yields

$$Q_2 = 4\pi B_{\nu a} K_a \rho_a$$

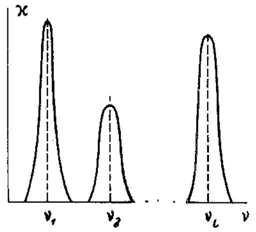

Fig. 4

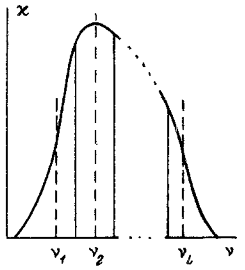

Fig. 5

The difference $Q_1 - Q_2$ gives the required quantity:

$$Q = \int_0^{2\pi}\int_0^{\pi} \Delta J_\nu(0)\,|\,m\,|^{-1} \sin\vartheta\, d\vartheta\, d\varphi - 4\pi B_\nu K\rho \tag{4.1}$$

where all the variables are taken at a , $\Delta J_\nu\ (0)$ is given by Eq. (3.4) or (3.5), and the frequency at a is taken from the equation $\nu_0 = n(p, T)$.

5. As an example, let us consider at a given instant the flow of gas in unbounded, spherically symmetric space. The meridional plane of this flow is shown in Fig. 3, where the distance r between a and the center of symmetry O is r_0. The axis of the local set of spherical coordinates s, θ , φ coincides with the straight line Oa.

At a given time t we know the functions p(r) and T(r), and hence, using Eq. (1.1), we know $\rho(r)$ as well. Let us determine $\tau_1 = \tau_\nu^+\ (0, \infty)$ for $\frac{1}{2}\pi \le \theta \le \pi$, by means of Eq. (2.4). The quantities K, ρ , $\partial n/\partial T$, $\partial n/\partial p$ depend only on r and t, but are independent of θ and φ. At a given time the derivatives along the direction ab in Eq. (2.3) are

$$\frac{\partial T}{\partial s} = \frac{\partial T(r, t)}{\partial r}\frac{dr}{ds}, \qquad \frac{\partial p}{\partial s} = \frac{\partial p(r, t)}{\partial r}\frac{dr}{ds}$$

where dr/ds is the derivative along ab. It follows from Fig. 3 that

$$r = \sqrt{s^2 + 2r_0 s\cos\vartheta + r_0^2}$$

so that

$$\left.\frac{dr}{ds}\right|_{r=r_0} = \left.\frac{s + r_0\cos\vartheta}{\sqrt{s^2 + 2r_0 s\cos\vartheta + r_0^2}}\right|_{r=r_0} = -\cos\vartheta \tag{5.1}$$

and

$$\tau_1 = -K\rho\,\mathrm{sign}\left(\frac{dn}{dr}\right)\left(\frac{\partial n}{\partial T}\frac{\partial T}{\partial r} + \frac{\partial n}{\partial p}\frac{\partial p}{\partial r}\right)^{-1}\sec\vartheta \qquad (r = r_0)$$

We emphasize that at time t all the quantities other than sec θ are functions of r alone. It is therefore convenient to write

$$\tau_1 = -N(r)\sec\vartheta, \quad N(r) = K\rho\left|\frac{dn}{ds}\right|^{-1} \quad \left(\frac{\pi}{2} \leqslant \vartheta \leqslant \pi\right) \tag{5.2}$$

Let us determine τ_3 using Eq. (3.3) and the expression for m_a in the form (2.7). When $\frac{1}{2}\pi \le \theta \le \pi$ we have

$$\mathrm{sign}\frac{dn}{ds} = -\,\mathrm{sign}\frac{dn}{dr}$$

and the result is

$$\tau_3 = \tau_1 \qquad (\tfrac{1}{2}\pi \leqslant \vartheta \leqslant \pi)$$

The simplicity of this equation lies in that the acute angles between the chord ab and the tangent to the circle $r = r_0$ at the points a and b are equal in absolute magnitude.

Let us now determine Q. Using Eqs. (2.7), (5.1), (3.5), and (4.1), remembering that $\Delta J_\nu\ (0) = 0$, if $\theta \in [0, \pi/2)$ and substituting $z = -\sec\ \theta$, we find that

$$\int_0^{\pi} \Delta J_\nu(0)\,|\,m\,|^{-1}\sin\vartheta\, d\vartheta = -\left|\frac{\partial n}{\partial T}\frac{\partial T}{\partial r} + \frac{\partial n}{\partial p}\frac{\partial p}{\partial r}\right| B_\nu \int_{\pi/2}^{\pi} (1 - e^{N\sec\vartheta})^2 \cos\vartheta\,\sin\vartheta\, d\vartheta$$

$$= B_\nu\left|\frac{dn}{dr}\right|\int_1^{\infty}(1 - 2e^{-Nz} + e^{-2Nz})\frac{dz}{z^3} = B_\nu\left|\frac{dn}{dr}\right|\left[\frac{1}{2} - 2E_3(N) + E_3(2N)\right].$$

where

$$E_q(R) = \int_1^\infty e^{-Rz} \frac{dz}{z^q}, \quad qE_{q+1}(R) = e^{-R} - RE_q(R)$$

The function $E_1(R) = Ei(R)$ is discussed in [3] and the function N is given by Eq. (5.2).

The integral with respect to θ depends only on r and thus is independent of φ. It follows that integration with respect to φ in Eq. (4.1) results in a factor of 2π and hence, finally,

$$Q = 2\pi B_\nu \left| \frac{dn}{dr} \right| \left[\frac{1}{2} - 2E_3(N) + E_3(2N) \right] - 4\pi B_\nu K \rho \tag{5.3}$$

where all the quantities are determined at given r and, moreover

$$\frac{dn}{dr} = \frac{\partial n}{\partial T} \frac{\partial T}{\partial r} + \frac{\partial n}{\partial p} \frac{\partial p}{\partial r}, \quad N = K\rho \left| \frac{dn}{dr} \right|^{-1}$$

In the energy equation for the system, which is given by Eq. (1.1), the quantity Q is now a function of r, as given by Eq. (5.3).

6. If there is a number of emission and absorption lines in the spectrum of the gaseous medium (Fig. 4), the absorption coefficient can be written in the form

$$\varkappa = \sum_{l=1}^{L} K_l(p, T)\, \delta[\nu - \nu_l(p, T)]$$

where K_l (p, T) is the area between the corresponding part of the graph of $\varkappa(\nu)$ and the $\varkappa = 0$ axis. The values of ν_l will determine the abscissa of the centers of gravity of these areas.

The function $\varkappa(\nu)$ shown in Fig. 5 (noncoherent emission) can also be represented approximately in this form.

At point a, the pressure and temperature are p and T, and the corresponding frequencies are $\nu_1(p, T), \nu_2(p, T), \ldots, \nu_L(p, T)$. The gas at a can absorb and emit energy at these frequencies. Let us consider the surfaces $\nu = \nu_{1a}, \nu = \nu_{2a}, \ldots, \nu = \nu_{La}$. Suppose that a ray defined by θ, φ cuts these surfaces at points $b_1, b_2, \ldots, b_L$. For each of these frequencies, surfaces with other frequency values will be optically transparent. This means that at each frequency the intensity loss $\Delta J_{\nu l}$ (0) due to absorption at a can be calculated as before from Eqs. (3.5) or (3.4), where b is the point of intersection of the straight line ac with the surface on which ν has the corresponding value $\nu_{la} = \nu_l(p_a, T_a)$.

As an example, when $J_\nu(S) = 0$, we have

$$\Delta J_{\nu l}(0) = B_{\nu l b}(1 - e^{-\tau_{1l}})(1 - e^{-\tau_{3l}})$$

$$\tau_{1l} = (|m_l| K_l \rho)_b, \quad \tau_3 = (|m_l| K_l \rho)_a$$

$$m_l = \left(\frac{\partial \nu_l}{\partial T} \frac{\partial T}{\partial s} + \frac{\partial \nu_l}{\partial p} \frac{\partial p}{\partial s} \right)^{-1}$$

where the subscripts ν_l correspond to the frequency ν_l, and a and b denote the corresponding points.

The loss ΔJ (0) due to absorption at a at all frequencies is given by the simple sum

$$\Delta J(0) = \Delta J_{\nu 1}(0) + \Delta J_{\nu 2}(0) + \ldots + J_{\nu L}(0)$$

The radiant-energy flux is now given by a formula analogous to Eq. (4.1):

$$Q = \sum_{l=1}^{L} \int_0^{2\pi} \int_0^{\pi} \Delta J(0) |m|^{-1} \sin\vartheta\, d\vartheta\, d\varphi - 4\pi\rho \sum_{l=1}^{L} B_{\nu l} K_l$$

but if there are a number of surfaces with the same value of ν_l we must sum over these surfaces as well.

Note Added in Proof. The value of Q_2 given on p. 14 is incorrect. In fact, the amount of energy emitted per unit volume is

$$Q_2 = 2\pi B_{\nu a}|\nabla n\, [1 - 2E_3 (\rho K\, |\nabla n\,|^{-1})]$$

This result can be obtained by analogy with the case of the radiation emitted by a thin layer.

LITERATURE CITED

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